Study of Spectral Statistics of Classically Integrable Systems

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In this work we present the results of a study of spectral statistics for a classically integrable system, namely the rectangle billiard. We show that the spectral statistics are indeed Poissonian in the semiclassical limit for almost all such systems, the exceptions being the atypical rectangles with rational squared ratio of its sides, and of course the energy ranges larger than $L_{\text{max}} = \hbar/T_0$, where T_0 is the period of the shortest periodic orbit of the system, however $L_{\text{max}} \to \infty$ when $E \to \infty$.

§1. Introduction

The quantal energy spectra of classically integrable and classically chaotic systems show a remarkable difference in their statistical properties. The former appear to be random, satisfying Poissonian statistics, while the latter behave as eigenvalues of random orthogonal (Gaussian orthogonal ensembles or GOE) or Hermitian matrices (Gaussian unitary ensembles, GUE), depending on the properties of the system with respect to antiunitary transformations like the time reversal ^{1), 2), 3)}.

These observations have, however, not yet been proven. There exists numerical evidence supporting the Poissonian model for the integrable case $^{4)}$. However, there has also been some evidence of the discrepancy of the Poissonian model for the case of the rectangle billiard $^{5), 6), 7)$.

In this work we study the spectral statistics of a few integrable systems, namely the rectangle, torus and circle billiards. Here we present only the results for the rectangle billiard. For results on the torus and circle billiards and a more in-depth discussion of the rectangle billiard see reference ⁸⁾.

§2. Definitions

We deal with an ordered spectrum $\{E_1, E_2, ..., E_i, ...\}$ of a given system. In order to compare spectral statistical properties of different systems it is convenient to eliminate the system dependent average density of states. This is done by a nonuniform mapping of the energy scale called the *unfolding of spectra*. If we know the average number of states up to a given energy $\overline{N}(E)$, which can be obtained in the first approximation by the Thomas-Fermi rule, then the mapping

$$x_i = \overline{N}(E_i) \tag{2.1}$$

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gives the unfolded spectrum $\{x_i\}$ of the system whose mean level spacing is equal to 1.

The first measure of statistical spectral properties is the nearest neighbour level spacing distribution P(S), which gives the distribution of the distances between the consecutive energy levels. For Poissonian systems it is (after unfolding)

$$P_{\text{Poisson}}(S) = \exp(-S).$$
 (2.2)

The next measure of the spectral properties is the Δ statistics. It is defined as

$$\Delta(L) = \frac{1}{L} \langle \min_{A,B} \int_{\alpha}^{\alpha + L} [N(x) - Ax - B] dx \rangle_{\alpha},$$
(2.3)

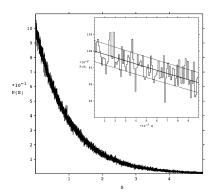
where N(x) is the spectral staircase function and counts the number levels up to energy E. This measure shows the average deviations of the spectral staircase function from the best fitting straight line over the interval L, and is therefore sometimes referred to as spectral rigidity. For systems with degeneracies of zero measure, for small L it is always

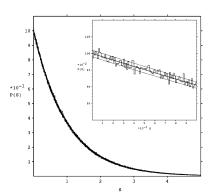
$$\Delta(L) = L/15. \tag{2.4}$$

For Poissonian spectra, however, this is true for all L.

Another spectral measure is important since all other measures can be derived from it. It is the E(k,L) distributions, which give the probability of finding exactly k levels in the interval of the length L. Due to the unfolding of spectra, their maximum is close to $L \approx k$. For Poissonian systems,

$$E_{\text{Poisson}}(k, L) = \frac{L^k}{k!} \exp(-L).$$
 (2.5)





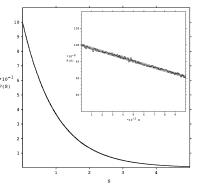


Fig. 1. The level spacing distribution for different energy windows, top to bottom: 10^6 levels above unfolded energy 10^9 (energy window O_1), 10^7 levels above energy 10^{10} (O_2) and 10^8 levels above energy 10^{11} (O_3). In the insert the appropriate magnifications for small level spacings are shown.

The GOE/GUE expressions for the E(k, L) statistics can be found in the form of tables in the reference 9) for k = 0 - 7, while for larger k the asymptotic Gaussian formulae are applicable ¹⁰⁾.

§3. Results

The billiard we studied was the rectangle. Its spectrum is in appropriate units given as

$$E_{m,n} = m^2 + \alpha n^2, \tag{3.1}$$

where m, n are positive integers and α is the squared ratio of sides.

If α is rational,

$$\alpha = \frac{p}{q},\tag{3.2}$$

where p and q are positive integers, the energy levels become integer multiples of the quantity $^{4)}$

$$X_{p,q} = \frac{\pi}{4\sqrt{pq}} \tag{3.3}$$

in the units of the mean level spacing. This means that the level spacing distribution becomes a sum of delta functions separated by these steps. In reference 8) we show that even the amplitudes for the delta peaks do not follow the Poissonian prediction and that the degeneracy of levels increases with energy. Connors and Keating 11) proved that indeed the level spacing distribution for the case of the square $\alpha=1$ billiard goes towards

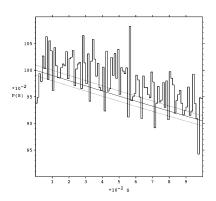
$$P_{\alpha=1}(S) = \delta(S), \tag{3.4}$$

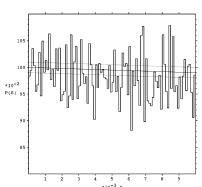
with $E \to \infty$.

Rational α are, however, not typical. For our studies we chose the irrational $\alpha = \frac{\pi}{3}$, but also verified the results with the value of $\alpha = \frac{\sqrt{5}-1}{2}$, which is the golden mean and which gave qualitatively the same results.

In the figure 1 we plot the level spacing distribution for different high lying energy windows. We see that the Poissonian statistics are followed excellently, with the statistical deviations remaining within the expected values.

If, however, we try to include all the levels up to a given energy E in the plot,





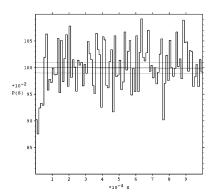


Fig. 2. P(S) for small level spacings for all levels up to energy (top to bottom) 10^7 , 10^8 and 10^9 , with the bin size chosen such that the expected number of events per bin is about 1000.

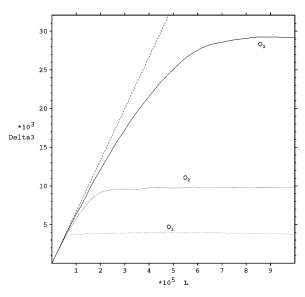


Fig. 3. The Δ statistics for the same energy windows as in the figure 1.

and shrink the bin sizes so that at all energies there is the same expected number of events in the bin, we can see in the figure 2 that there exist fluctuations around the Poissonian value which are larger than statistically expected and which do not seem to decrease with energy. We show in reference 8) that this happens due to the sufficiently influential rational α close to the chosen irrational value. If the bin size is kept constant, however, the distribution fluctuations tend to the statistically expected ones.

The long range Δ statistics also behaves as expected, as can be seen in the figure 3. For $L < L_{\text{max}}$, where $L_{\text{max}} = \hbar/T_0$ with T_0 being the period of the shortest periodic orbit, Δ follows the Poissonian prediction. Berry ¹²⁾ predicted through the use of the periodic orbit theory that at larger L the Δ should saturate, as can also be confirmed in our plot. We may notice, however, that $L_{\text{max}} \propto \sqrt{E} \to \infty$ as $E \to \infty$.

The E(k,L) measures for different k are plotted in figure 4. We can see that the Poissonian prediction is nicely followed up to k being a fraction of the $L_{\rm max}$, where the numerical functions become narrower than the Poissonian ones. This can be understood by noting through the Δ plots that the spectrum becomes more rigid at higher L than expected for the Poissonian one, so we should expect exactly k levels to be found in a smaller ranges of L around the average value $\langle L \rangle = k$. But, as $L_{\rm max}$ goes to infinity with increasing energy, the Poissonian E(k,L) statistics are followed for all k and L in our system in the limit $E \to \infty$.

§4. Conclusion

The Poissonian statistics are indeed nicely satisfied in the systems we studied, apart from the atypical cases of the rational squared ratio of sides of the rectangle, and the energy ranges greater than L_{max} . We show the same to be true also for the

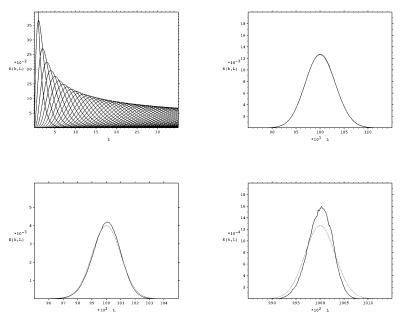


Fig. 4. The E(k, L) statistics for the energy window 10^8 levels above unfolded energy 10^{11} and k = 0..50 (top left), $k = 10^3$ (top right), $k = 10^4$ (bottom left) and $k = 10^5$ (bottom right)

cases of the torus and circle billiards in reference 8).

These studies are not important to understand the properties of the integrable systems only. In the case of the most general, mixed type systems, the total spectrum is in the semiclassical limit composed of independent contributions of the regular and chaotic components, with the regular component contribution having Poissonian and the chaotic component contribution the GOE(GUE) statistics (the Berry-Robnik ¹³⁾ theory), so a good understanding of the individual components is necessary in order to understand the whole picture. For a study of mixed type systems see reference 14) and the short version in reference 15)

References

- [1] O. Bohigas, M.-J. Giannoni and C. Schmit, Phys. Rev. Lett. 52 (1984), 1
- [2] M. V. Berry and M. Robnik, J. Phys. A: Math. Gen. 19 (1986), 669
- [3] M. Robnik, Lect. Notes Phys. **263** (1986), 120
- [4] M. V. Berry and M. Tabor, Proc. R. Soc. A156 (1977), 375
- [5] G. Casati, B. V. Chirikov and I. Guarneri, Phys. Rev. Lett. **54** (1985), 1350
- [6] T. H. Seligman and J. J. Verbaarschot, Phys. Rev. Lett. 56 (1986), 2767
- M. Feingold, Phys. Rev. Lett. **55** (1985), 2626
- [8] M. Robnik and G. Veble, J. Phys. A: Math. Gen. 31 (1998), 4669
- [9] M. L. Mehta, Random Matrices (San Diego: Academic, 1991).
- [10] R. Aurich, A. Bäcker and F. Steiner, Int. J. Mod. Phys. 11 (1997), 805
- 11 R. D. Connors and J. P. Keating, J. Phys. A: Math. Gen. 30 (1997), 1817
- [12] M. V. Berry, Proc. R. Soc. A400 (1985), 229
- [13] M. V. Berry and M. Robnik, J. Phys. A: Math. Gen. 17 (1984), 2413
- [14] G. Veble, M. Robnik and J. Liu, J. Phys. A: Math. Gen. 32 (1999), 6423
- [15] G. Veble, M. Robnik and J. Liu, Study of Regular and Irregular States in Generic Systems, this volume.